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ON THE WEAK SOLUTION OF COMPOUND LAPLACE BESSEL EQUATION

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ABSTRACT

In this paper, we have studied the compound Laplace Bessel equation of

$$\sum_{r=0}^{m} c_r \Delta_B^r u(x) = f(x)$$

where Δ_B^r is the Laplace – Bessel operator iterated r-times, $f(x)$ is a given generalized function, $u(x)$ is an unknown function, $x \in \mathbb{R}_n^+$ and c_r is a constant. In this work, we study the weak solution $u(x)$ of above the equation.

KEYWORDS: Weak solution, Compound Laplace Bessel equation, Tempered distribution.

I. INTRODUCTION

Sarikaya and Yildirim [3] have shown that the n-dimensional classical Laplace-Bessel equation have $u(x) = (-1)^k R_{2k}^e(x)$ is an elementary solution of the equation $\Delta_B^k u(x) = \delta$. Later, Kananthai and Nonlaopon [2] have studied the weak solution of the compound ultra-hyperbolic equation. Sarikaya and Yildirim [5] have studied the weak solution of the compound Bessel ultra-hyperbolic equation. Moreover, Bupasiri and Nonlaopon [6] have studied the weak solution of compound equations related to the ultra-hyperbolic operators. In this article, we will consider the Laplace-Bessel operator iterated k-time with $x \in \mathbb{R}_n^+ = \{x : x = (x_1,\ldots,x_n), x_1 > 0,\ldots,x_n > 0\}$

$$\Delta_B^k = \left(\sum_{i=1}^{p} B_{x_i} + \sum_{j=p+1}^{p+q} B_{x_j} \right)^k, \quad p+q=n \tag{1.1}$$

$B_{x_i} = \dfrac{\partial^2}{\partial x_i^2} + \dfrac{2v_i}{x_i}\dfrac{\partial}{\partial x_i}$ where $2v_i = 2\alpha_i + 1$, $2\alpha_i > -\dfrac{1}{2}$, $x_i > 0$, $i = 1,2,\ldots,n$, k is nonnegative integer and n is dimension of $x \in \mathbb{R}_n^+$. Consider the equation

$$\Delta_B^k u(x) = f(x) \tag{1.2}$$

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where $u(x)$ and $f(x)$ are some generalized functions. We will develop the equation (1.2) to the form

$$\sum_{r=0}^{m} c_r \Delta_B^r u(x) = f(x) \tag{1.3}$$

which is called the compound Laplace Bessel and by convention $\Delta_B^0 u(x) = u(x)$. In finding the solutions of (1.3), we use the properties of convolutions for the generalized functions.

II. PRELIMIMARIES

Definition 2.1. Let $x = (x_1, \ldots, x_n)$ be a point of the n - dimensional space $\mathbb{R}_n^+$

$$V = x_1^2 + x_2^2 + \cdots + x_p^2 + x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2 \tag{2.1}$$

where $p + q = n$. For any complex number α, defined the function

$$R_\alpha^e(x) = \frac{V^{\frac{\alpha - n - 2|\nu|}{2}}}{w_n(\alpha)} \tag{2.2}$$

where
$$w_n(\alpha) = \frac{\prod_{i=1}^{n} 2^{\nu_i - \frac{1}{2}} \Gamma\left(\nu_i + \frac{1}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)}{2^{n+2|\nu|-2\alpha} \Gamma\left(\frac{n + 2|\nu| - \alpha}{2}\right)}, \quad |\nu| = \nu_1 + \cdots + \nu_n. \tag{2.3}$$

The function $R_\alpha^e(x)$ is call the *elliptic kernel of Marcel Riesz* and an ordinary function if $\operatorname{Re}\alpha \ge n$ and is a distribution of α if $\operatorname{Re}\alpha < n$.

Lemma 2.1. *$R_{2k}^e(x)$ is a homogeneous distribution of order $(2k - n - 2|\nu|)$. In particular, it is a tempered distribution.*

Proof. We need to show that $R_{2k}^e(x)$ satisfies the Euler equation

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_{2k}^e(x) = (2k - n - 2|\nu|) R_{2k}^e(x).$$

Now
$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_{2k}^e(x) = \frac{1}{w_n(2k)} \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \left(x_1^2 + \cdots + x_p^2 + x_{p+1}^2 + \cdots + x_{p+q}^2\right)^{\frac{2k-n-2|\nu|}{2}}$$

$$= \frac{1}{w_n(2k)} (2k - n) \left(x_1^2 + \cdots + x_p^2 + x_{p+1}^2 + \cdots + x_{p+q}^2\right)^{\frac{2k-n-2|\nu|}{2} - 1}$$
$$\times \left(x_1^2 + \cdots + x_p^2 + x_{p+1}^2 + \cdots + x_{p+q}^2\right)$$

$$= \frac{1}{w_n(2k)} (2k - n) \left(x_1^2 + \cdots + x_p^2 + x_{p+1}^2 + \cdots + x_{p+q}^2\right)^{\frac{2k-n-2|\nu|}{2}}$$

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$$= \frac{(2k-n-2|v|)V^{\frac{2k-n-2|v|}{2}}}{w_n(2k)}$$

$$= (2k-n-2|v|)R^e_{2k}(x).$$

Hence $R^e_{2k}(x)$ is a homogeneous distribution of order $(2k-n-2|v|)$. Donoghue [7] prove that every homogeneous distribution is a tempered distribution. So $R^e_{2k}(x)$ is a tempered distribution. This is complete of proof.

Lemma 2.2. *Given the equations*

$$\Delta^k_B u(x) = \delta(x) \tag{2.4}$$

where Δ^k_B *is defined by* (1.1), $x \in \mathbb{R}^+_n$ *and* $\delta(x)$ *is the Dirac-delta distribution, then we obtain* $u(x) = (-1)^k R^e_{2k}(x)$ *as an elementary solution of* (2.4), *where* $R^e_{2k}(x)$ *is defined by* (2.2) *with* $\alpha = 2k$.

Proof. See [3].

Lemma 2.3. *(The convolutions of tempered distributions)*

(a) $\left(\Delta^k_B \delta\right) * u(x) = \Delta^k_B u(x)$ *where* $u(x)$ *is any tempered distribution.*

(b) Let $R^e_{2k}(x)$ *and* $R^e_{2m}(x)$ *be defined by* (2.2), *then* $R^e_{2k}(x) * R^e_{2m}(x)$ *exists and is a tempered distribution.*

(c) Let $R^e_{2k}(x)$ *and* $R^e_{2m}(x)$ *be defined by* (2.2), *then* $R^e_{2k}(x) * R^e_{2m}(x) = R^e_{2k+2m}(x)$ *where* k *and* m *are nonnegative integer.*

(d) Let $R^e_{2k}(x)$ *and* $R^e_{2m}(x)$ *be defined by* (2.2) *and if* $R^e_{2k}(x) * R^e_{2m}(x) = \delta$ *then* $R^e_{2k}(x)$ *is an inverse of* $R^e_{2m}(x)$ *in the convolution algebra, denoted by* $R^e_{2k}(x) = R^{e*-1}_m(x)$.

Proof. (a) First, we consider the case $k = 1$, now

$$\Delta_B \delta(x) = \left(\sum_{i=1}^{p} \frac{\partial^2 \delta(x)}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial \delta(x)}{\partial x_i} \right) + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \delta(x)}{\partial x_j^2} + \frac{2v_j}{x_j} \frac{\partial \delta(x)}{\partial x_j} \right), \; p+q=n$$

and let $\varphi(x)$ be a testing function in the Schwarts space S. By the definition odd B-convolution, we have

$$\left\langle \Delta_B \delta(x) * u(x), \varphi(x) \right\rangle = \langle u(x), \langle \Delta_B \delta(x), \varphi(x+y) \rangle \rangle$$

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$$=\left\langle u(x),\left\langle\left(\sum_{i=1}^{p}\frac{\partial^2\delta(y)}{\partial x_i^2}+\frac{2v_i}{x_i}\frac{\partial\delta(y)}{\partial x_i}\right)+\left(\sum_{j=p+1}^{p+q}\frac{\partial^2\delta(y)}{\partial x_j^2}+\frac{2v_j}{x_j}\frac{\partial\delta(y)}{\partial x_j}\right),\varphi(x+y)\right\rangle\right\rangle$$

$$=\left\langle u(x),\left\langle \delta(y),\left(\sum_{i=1}^{p}\frac{\partial^2\varphi(x+y)}{\partial x_i^2}+\frac{2v_i}{x_i}\frac{\partial\varphi(x+y)}{\partial x_i}\right)+\left(\sum_{j=p+1}^{p+q}\frac{\partial^2\varphi(x+y)}{\partial x_j^2}+\frac{2v_j}{x_j}\frac{\partial\varphi(x+y)}{\partial x_j}\right)\right\rangle\right\rangle$$

$$=\left\langle u(x),\left(\sum_{i=1}^{p}\frac{\partial^2\varphi(x)}{\partial x_j^2}+\frac{2v_i}{x_i}\frac{\partial\varphi(x)}{\partial x_i}\right)+\left(\sum_{j=p+1}^{p+q}\frac{\partial^2\varphi(x)}{\partial x_j^2}+\frac{2v_j}{x_j}\frac{\partial\varphi(x)}{\partial x_j}\right)\right\rangle$$

$$=\left\langle\left(\sum_{i=1}^{p}\frac{\partial^2 u(x)}{\partial x_j^2}+\frac{2v_i}{x_i}\frac{\partial u(x)}{\partial x_i}\right)+\left(\sum_{j=p+1}^{p+q}\frac{\partial^2 u(x)}{\partial x_j^2}+\frac{2v_j}{x_j}\frac{\partial u(x)}{\partial x_j}\right),\varphi(x)\right\rangle$$

$$=\langle\Delta_B u(x),\varphi(x)\rangle .$$

It follows that

$$\Delta_B\delta(x)*u(x)=\Delta_B u(x).$$

Similarly for any k , we can show that

$$\Delta_B^k\delta(x)*u(x)=\Delta_B^k u(x).$$

(b) By Lemma 2.1, thus $R_{2k}^e(x)$ and $R_{2m}^e(x)$ are tempered distribution and $R_{2k}^e(x) * R_{2m}^e(x)$ exists and is a tempered distribution by [4].

(c) From equation $\Delta_B^{k+m}u(x)=\delta(x)$ we obtain $u(x)=(-1)^{k+m}R_{2k+2m}(x)$ by Lemma 2.2. For any m is a nonnegative integer, we write

$$\Delta_B^{k+m}u(x)=\Delta_B^k\Delta_B^m u(x)=\delta(x)$$

then by Lemma 2.2 we have the following equality

$$\Delta_B^m u(x)=(-1)^k R_{2k}^e(x).$$

Convolving both sides by $(-1)^m R_{2m}^e(x)$ we obtain

$$(-1)^m R_{2m}^e(x)*\Delta_B^m u(x)=(-1)^k R_{2k}^e(x)*(-1)^m R_{2m}^e(x)$$

or

$$\Delta_B^m\left((-1)^m R_{2m}^e(x)\right) * u(x) = (-1)^{k+m} R_{2k}^e(x) * R_{2m}^e(x).$$

Then from Lemma 2.2 we have the following equality

$$\delta(x) * u(x) = (-1)^{k+m} R_{2k}^e(x) * R_{2m}^e(x).$$

It follows that

$$u(x) = (-1)^{k+m} R_{2k}^e(x) * R_{2m}^e(x).$$

From the fact that $u(x) = (-1)^{k+m} R_{2k+2m}^e(x)$ we obtain $R_{2k}^e(x) * R_{2m}^e(x) = R_{2k+2m}^e(x)$.

(d) Since $R_{2k}^e(x)$ and $R_{2m}^e(x)$ are tempered distributions with compact supports,

thus $R_{2k}^e(x)$ and $R_{2m}^e(x)$ are the elements of space of convolution algebra u' of distribution. Now

$R_{2k}^e(x) * R_{2m}^e(x) = \delta(x)$ then by Zemanian [1] show that $R_{2k}^e(x) = R_{2m}^{e*-1}(x)$ is an inverse.

Lemma 2.4. *Let* $R_{2k}^e(x)$ *and* $w_n(2k)$ *, be defined by* (2.2) *and* (2.3). *Then*

(a) $w_n(2k+2) = 8k(n+2|v|-2k-2)w_n(2k)$.

(b) $\Delta_B^k R_{2m}^e(x) = (-1)^k R_{2m-2k}^e(x)$, *where k and m are nonnegative integer.*

(c) $R_{-2k}^e(x) = (-1)^k \Delta_B^k \delta(x)$, *where k is a nonnegative integer.*

Proof. (a) From (2.3), we have

$$w_n(2k+2) = \frac{\prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right)\Gamma(k+1)}{2^{n+2|v|-4k-4}\Gamma\left(\frac{n+|v|-2k-2}{2}\right)}$$

$$= \frac{8k(n+2|v|-2k-2)\prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right)\Gamma(k)}{2^{n+2|v|-4k}\Gamma\left(\frac{n+|v|-2k-2}{2}\right)}$$

$$= 8k(n+2|v|-2k-2)w_n(2k).$$

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(b) By Lemma 2.3(c), we have

$$\delta * R^e_{2m}(x) = R^e_{2k}(x) * R^e_{2m-2k}(x)$$

$$\Delta^k_B (-1)^k R^e_{2k}(x) * R^e_{2m}(x) = R^e_{2k}(x) * R^e_{2m-2k}(x)$$

$$(-1)^k R^e_{2k}(x) * \Delta^k_B R^e_{2m}(x) = R^e_{2k}(x) * R^e_{2m-2k}(x)$$

and

$$\Delta^k_B R^e_{2m}(x) = (-1)^k R^e_{2m-2k}(x).$$

(c) For $m = k$ by Lemma 2.4(b) we have

$$\Delta^k_B R^e_{2m}(x) = (-1)^k R^e_0(x), \quad R^e_0 = \delta .$$

For $m = 0$, by Lemma 2.4(b) we have $\Delta^k_B R^e_0(x) = (-1)^k R^e_{-2k}(x)$ or $(-1)^k \Delta^k_B \delta = R^e_{-2k}(x)$.

III. MAIN RESULTS

Theorem 3.1. *Given the compound Laplace – Bessel equation*

$$\sum_{r=0}^{m} C_r \Delta^r_B u(x) = f(x) \tag{3.1}$$

where Δ^r_B *is the Laplace – Bessel operator iterated* k *-times defined by* (1.1), $f(x)$ *is a given generalized function,* $u(x)$ *is an unknown function,* $x \in \mathbb{R}^+_n$ *and* C_r *is a constant. Then* (3.1) *has a weak solution*

$$u(x) = f(x) * R^e_{2m}(x) * \left((-1)^m C_m R^e_0(x) + w(x) R^e_2(x) \right)^{*-1} \tag{3.2}$$

where

$$w(x) = (-1)^{m-1} C_{m-1} + (-1)^{m-2} C_{m-2} \frac{V}{8(n+2|v|-4)} + (-1)^{m-3} C_{m-3} \frac{V^2}{8 \cdot 16 (n+2|v|-4)(n+2|v|-6)}$$

$$\cdots + C_0 \frac{V^{m-1}}{8 \cdot 16 \cdot 24 \cdots 8(m-1)(n+2|v|-4)(n+2|v|-6)\cdots(n+2|v|-2m)} \tag{3.3}$$

and V *defined by* (2.1) *and* $\left((-1)^m C_m R^e_0(x) + w(x) R^e_2(x) \right)^{*-1}$ *is an inverse of*

$$(-1)^m C_m R_0^e(x) + w(x) R_2^e(x).$$

Proof. By Lemma 2.3(a), equation (3.1) can be written as

$$(C_m \Delta_B^m \delta + C_{m-1} \Delta_B^{m-1} \delta + \cdots + C_1 \Delta_B \delta + C_0 \delta) * u(x) = f(x).$$

Convolving both sides by $R_{2m}^e(x)$ defined by (2.2), we obtain

$$\left(C_m \Delta_B^m R_{2m}^e(x) + C_{m-1} \Delta_B^{m-1} R_{2m}^e(x) + \cdots + C_1 \Delta_B R_{2m}^e(x) + C_0 R_{2m}^e(x) \right) * u(x) = f(x) * R_{2m}^e(x).$$

By Lemma 2.2 and Lemma 2.4(b), we obtain

$$\left((-1)^m C_m \delta + (-1)^{m-1} C_{m-1} R_2^e(x) + \cdots + (-1) C_1 R_{2(m-1)}^e(x) + C_0 R_{2m}^e(x) \right) * u(x) = f(x) * R_{2m}^e(x). \qquad (3.4)$$

By Lemma 2.4(a), we obtain $R_4^e(x) = \dfrac{V^{\frac{4-n-2|v|}{2}}}{w_n(4)} = R_2^e(x) \cdot \dfrac{V}{8(n+2|v|-4)}$.

Similarly,

$$R_6^e(x) = R_2^e(x) \cdot \frac{V^2}{8 \cdot 16 (n+2|v|-4)(n+2|v|-6)}$$

$$R_8^e(x) = R_2^e(x) \cdot \frac{V^3}{8 \cdot 16 \cdot 24 (n+2|v|-4)(n+2|v|-6)(n+2|v|-8)}$$

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$$R_{2m}^e(x) = R_2^e(x) \cdot \frac{V^{m-1}}{8 \cdot 16 \cdot 24 \cdots 8(m-1)(n+2|v|-4)(n+2|v|-6) \cdots (n+2|v|-2m)}.$$

Thus we obtain the function $w(x)$ of (3.3). Now $w(x)$ is continuous and infinitely differentiable in classical sense for n is odd. Since $R_2^e(x)$ is a tempered distribution with compact support, hence $w(x) R_2^e(x)$ also is tempered distribution with compact support and so $(-1)^m C_m R_0^e(x) + w(x) R_2^e(x)$. By Lemma 2.3(d), $(-1)^m C_m R_0^e(x) + w(x) R_2^e(x)$ has an inverse denote by

$$\left((-1)^m C_m R_0^e(x) + w(x) R_2^e(x) \right)^{*-1}.$$

Now (3.4) can be written as

$$\left((-1)^m C_m R_0(x) + w(x) R_2(x) \right) * u(x) = f(x) * R_{2m}(x), \quad R_0 = \delta.$$

Convolving both sides by

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$$\left((-1)^m C_m R_0^e(x) + w(x) R_2^e(x)\right)^{*-1},$$

we have

$$u(x) = f(x) * R_{2m}^e(x) * \left((-1)^m C_m R_0^e(x) + w(x) R_2^e(x)\right)^{*-1}$$

is weak solution of (3.1) with odd-dimensional n. This completes the proof.

Corollary 3.1. *Given the compound Laplace – Bessel equation*

$$\sum_{r=0}^{m} C_r \Delta_B^r u(x) = \delta(x) \tag{3.5}$$

where Δ_B^r *is the Laplace – Bessel operator iterated* r*-times defined by* (1.1), $\delta(x)$ *is a given Dirac delta distribution,* $u(x)$ *is an unknown function,* $x \in \mathbb{R}_n^+$ *and* C_r *is a constant. Then* (3.5) *has a solution*

$$u(x) = R_{2m}^e(x) * \left((-1)^m C_m R_0^e(x) + w(x) R_2^e(x)\right)^{*-1} \tag{3.6}$$

where

$$w(x) = (-1)^{m-1} C_{m-1} + (-1)^{m-2} C_{m-2} \frac{V}{8(n+2|v|-4)} + (-1)^{m-3} C_{m-3} \frac{V^2}{8 \cdot 16 (n+2|v|-4)(n+2|v|-6)}$$

$$+ \cdots + C_0 \frac{V^{m-1}}{8 \cdot 16 \cdot 24 \cdots 8(m-1)(n+2|v|-4)(n+2|v|-6)\cdots(n+2|v|-2m)} \tag{3.7}$$

and V *defined by* (2.1) *and* $\left((-1)^m C_m R_0^e(x) + w(x) R_2^e(x)\right)^{*-1}$ *is an inverse of*

$$(-1)^m C_m R_0^e(x) + w(x) R_2^e(x).$$

Proof. If $f(x) = \delta(x)$, then we have

$$u(x) = R_{2m}^e(x) * \left((-1)^m C_m R_0^e(x) + w(x) R_2^e(x)\right)^{*-1}$$

yielding the result, where $\delta(x)$ is Dirac delta distribution and $f(x)$ is generalized function.

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